PROPERTIES OF GENERALISED JUXTAPOLYNOMIALS

BY

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ABSTRACT

Given $F(z), f_1(z), f_2(z), \ldots, f_n(z)$ defined on a finite point set E, and given B — the set of generalised polynomials $\Sigma_{k=1}^{n} a_k f_k(z)$ — the definition of a juxtapolynomial is extended in the following manner: for a fixed $\lambda(0 < \lambda \leqq 1)$, $f(z) \in B$ is called a generalized λ-weak juxtapolynomial to $F(z)$ on E if and only if there exists no $g(z) \in B$ for which $g(z) = F(z)$ whenever $f(z) = F(z)$ and $|g(z) - F(z)| < \lambda |f(z) - F(z)|$ whenever $f(z) \neq F(z)$. The properties of such $f(z)$ are investigated with particular attention given to the real case.

1. Preliminaries. Throughout this paper, we assume E is a finite point set in the complex domain, consisting of at least $n+1$ points, $F(z), f_1(z), \cdots, f_n(z)$ are given complex functions defined on E, and $\Sigma_{k=1}^{n} a_k f_k(z) = 0$ on any n points of $E \Leftrightarrow a_1 = a_2 = \cdots = a_n = 0$, this condition which we denote by Γ, is obviously fulfilled by the functions $f_k(z) \equiv z^{n-k}$ $k = 1, 2, \cdots, n$. Let B be the class of all functions of the form $\Sigma_{k=1}^{n} a_k f_k(z)$ where $a_1, a_2, \cdots, a_n$ are constants. For any fixed λ, $0 < \lambda \leqq 1$, and any set $E' \subseteq E$, we define the following classes of functions:

$f(z) \in J_1(\lambda, E') \Leftrightarrow f(z) \in B$ and there is no $g(z) \in B$ which for every $z \in E'$ satisfies the inequality $|g(z) - F(z)| \overset{*}{<} \lambda |f(z) - F(z)|$ (the symbol $\overset{*}{<}$ means $g(z) = F(z)$ when $f(z) = F(z)$ and $|g(z) - F(z)| < \lambda |f(z) - F(z)|$ when $f(z) \neq F(z)$).

$f(z) \in J_2(\lambda, E') \Leftrightarrow f(z) \in B$ and there is no $g(z) \in B$ which for every $z \in E'$ satisfies the inequality $|g(z) - F(z)| \leqq \lambda |f(z) - F(z)|$. $J_1(\lambda, E')$ $[J_2(\lambda, E')]$ is called the class of generalised λ-weak [λ-strong] juxtapolynomials to $F(z)$ on E'.

Let $J_3(\lambda, E') = J_1(\lambda, E') - J_2(\lambda, E')$.

2. When $f_k(z) \equiv z^{n-k}$ $k = 1, 2, \cdots, n$, $J_1(1, E)$ is defined by T. S. Motzkin and J. L. Walsh as the class of *juxtapolynomials* of degree $n-1$ over E, namely — "nearest" polynomials to $F(z)$ on E — and if in addition $F(z) \equiv z^n$: $\{z^n - f(z); f(z) \in J_1(1, E)\}$ is known as the class of *infrapolynomials* of degree

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n over E. The importance of the class $J_1(1,E)$ is in the fact that only for such functions $f(z)$, a minimum deviation of a monotone norm is obtained: $M = M(f(z), F(z), E)$ is called a *monotone norm* if M is a positive function defined for $f(z) \in B$, which decreases when $f(z)$ is replaced by $g(z) \in B$ for which $|g(z) - F(z)| \overset{*}{<} |f(z) - F(z)|$ on E. Two well known examples of monotone norms are:

Tchebycheff norm: $M = \max\{|f(z) - F(z)|;\ z \in E\}$

Least pth power norm: $M = \sum_{z \in E} |f(z) - F(z)|^p$ where $p > 0$.

3. The basis of this work are papers [1], [2] by the late Prof. M. Fekete. By similar methods a generalization of his results is obtained, giving in the end the structure of a function $f(z)$ in $J_1(1,E)$ without the restriction $f(z) \neq F(z)$ throughout E. We shall also see (Theorems 1,4), that if $F(z) \notin B$ then $B = J_2(1,E) \bigcup_{\lambda<1} J_3(\lambda, E)$, which is a union of disjoint sets; also if $f(z) \in J_k(\lambda, E)$ ($k = 1, 2$, or 3), then there exists a set $E' \subseteq E$ of $2n+1$ points at most (and when $F(z)$, $f_j(z)$ $(1 \leqq j \leqq n)$, $f(z)$ are real E' contains $n+1$ points exactly) such that $f(z) \in J_k(\lambda, E')$. When $F(z)$, $f_j(z)$ $(1 \leqq j \leqq n)$, and $f(z) \in B$ are real we find a representation for $f(z)$ in $J_k(\lambda, E)$ $(k = 2, 3)$. Finally, some topological properties (connectedness, compactness) of $J_k(\lambda, E)$ $(k = 1, 2)$ are discussed for the real case.

Some of the results are easily extended to the case where E is an infinite compact set, and $F(z)$, $f_j(z)$ $(1 \leqq j \leqq n)$ are continuous on E. Furthermore, E is taken in the complex domain merely for convenience, and all the results concerning B are in no way altered if E is assumed to be any abstract finite point set.

LEMMA 1. $J_3(1,E) = \varnothing$.

Proof. From the definitions $J_2(1,E) \subseteq J_1(1,E)$. $B \neq J_1(1,E)$, because due to condition Γ we may construct $f_0(z) \in B$ distinct from zero on E, and $cf_0(z) \notin J_1(1,E)$ for sufficiently large constants c. Then if $f(z) \in B - J_2(1,E)$, there exists $(f(z) \not\equiv) g(z) \in B$ such that $|g(z) - F(z)| \leqq |f(z) - F(z)|$ on E. Let $h(z) \in B$ be any function which assumes the values $g(z) - F(z)$ on $E_0 = \{z \in E;\ f(z) = g(z)\}$; such $h(z)$ exist due to condition Γ and the fact that E_0 contains less than n points (otherwise $g(z) \equiv f(z)$!). Let $g_\varepsilon(z) \equiv \frac{1}{2}g(z) + \frac{1}{2}f(z) - \varepsilon h(z)$ where $0 < \varepsilon < 1$ will be determined later. For any $z_0 \in E$ we have one of the following cases:

(i) $z_0 \in E_0$. Then clearly $|g_\varepsilon(z_0) - F(z_0)| \overset{*}{<} |f(z_0) - F(z_0)|$.

(ii) $|g(z_0) - F(z_0)| < |f(z_0) - F(z_0)|$. Then

$$|\tfrac{1}{2}g(z_0) + \tfrac{1}{2}f(z_0) - F(z_0)| \leqq \tfrac{1}{2}|g(z_0) - F(z_0)| + \tfrac{1}{2}|f(z_0) - F(z_0)| < |f(z_0) - F(z_0|,$$

and since E finite, there exists ε_1 $(0 < \varepsilon_1 < 1)$ such that for every $0 < \varepsilon \leqq \varepsilon_1$ and every z_0 as above $|g_\varepsilon(z_0) - F(z_0)| < |f(z_0) - F(z_0)|$.

(iii) $z_0 \notin E_0$ and $|g(z_0) - F(z_0)| = |f(z_0) - F(z_0)|$. Then

$$\left|\tfrac{1}{2}g(z_0) + \tfrac{1}{2}f(z_0) - F(z_0)\right| < \tfrac{1}{2}|g(z_0) - F(z_0)| + \tfrac{1}{2}|f(z_0) - F(z_0)| = |f(z_0) - F(z_0)|,$$

and here too there is ε_2 $(0 < \varepsilon_2 \leqq \varepsilon_1)$ such that for any z_0 as above $|g_{\varepsilon_2}(z_0) - F(z_0)| < |f(z_0) - F(z_0)|$.

We proved that in all cases $|g_{\varepsilon_2}(z) - F(z)| \overset{*}{<} |f(z) - F(z)|$, that is $f(z) \notin J_1(1, E)$.

4. THEOREM 1. *If $F(z) \notin B$ then $B = J_2(1,E) \bigcup_{\lambda < 1} J_3(\lambda, E)$.*

Proof. Let $f(z) \in B - J_1(1,E)$ and let $\lambda' = \mathrm{Sup}\{\lambda; f(z) \in J_1(\lambda, E)\}$, then $\lambda' > 0$, since $\lambda' = 0$ implies that for every $\lambda \leqq 1$, there is $g_\lambda(z) \in B$ such that $|g_\lambda(z) - F(z)| \overset{*}{<} \lambda |f(z) - F(z)|$ on E; taking a sequence $\lambda_m \to 0$, we obtain a sequence $g_{\lambda_m}(z) \to F(z)$ on E, a suitable subsequence $g_{\lambda_{m_i}}(z)$ will converge to a limit function $g(z) \in B$; this is due to condition Γ, since if $M = \mathrm{Max}\{|f(z) - F(z)| + |F(z)|; z \in E\}$, the set $g_{\lambda_m}(z)$ is uniformly bounded by M on E, and if $\boldsymbol{a}^{(m)} = (a_1^{(m)}, a_2^{(m)}, \cdots, a_n^{(m)})$ is the coefficients' vector of $g_{\lambda_m}(z)$ then due to condition Γ for any set of distinct points $\{z_1, z_2, \cdots, z_n\} \subseteq E$, the vectors $\boldsymbol{f}(z_i) = \overline{(f_1(z_i)}, \overline{f_2(z_i)}, \cdots, \overline{f_n(z_i)})$ $i = 1, \cdots, n$ form a basis to the n-dimensional complex vector space, and as the scalar product $g_{\lambda_m}(z_i) = (\boldsymbol{a}^{(m)}, \boldsymbol{f}(z_i))$ is uniformly bounded, it follows that the set $\{\boldsymbol{a}^{(m)}\}$ is uniformly bounded and therefore has a subsequence $\{\boldsymbol{a}^{(m_i)}\}$ which converges to a limit point $\boldsymbol{a}' = (a_1', a_2', \cdots, a_n')$, for which

$$F(z) = \lim_{i \to \infty} g_{\lambda_{m_i}}(z) = g(z) \left(\equiv \sum_{k=1}^{n} a_k' f_k(z)\right) \text{ for every } z \in E,$$

a contradiction!

$f(z) \in J_1(\lambda', E)$, since assuming otherwise, there exists $g(z) \in B$ with $|g(z) - F(z)| \overset{*}{<} \lambda' |f(z) - F(z)|$ on E, E being finite, the inequality holds for some λ, $0 < \lambda < \lambda'$, contrary to the definition of λ'. Assume $f(z) \in J_2(\lambda', E)$, then there is (contrary to the definition of λ'!) $\varepsilon > 0$, $\varepsilon + \lambda' < 1$, such that $f(z) \in J_1(\lambda' + \varepsilon, E)$, because otherwise for every $\varepsilon > 0$, $\varepsilon + \lambda' < 1$, exists $g_\varepsilon(z) \in B$ with $|g_\varepsilon(z) - F(z)| \leqq (\lambda' + \varepsilon)|f(z) - F(z)|$ on E, and similarly as above taking a sequence $\varepsilon_m \to 0$ with $g_{\varepsilon_m}(z)$ converging to a limit function $g(z) \in B$ on E, we obtain the inequality $|g(z) - F(z)| \leqq \lambda' |f(z) - F(z)|$ on E, impossible by our assumption. Hence $f(z) \notin J_2(\lambda', E)$. Note that the classes which compose B are all distinct.

We see now that $f(z) \in J_2(\lambda, E) \Leftrightarrow \lambda < \lambda'$. We shall show later how λ' may be calculated under certain conditions (Theorem 4). Lemmas 2, 3 are required in the proofs of some later results.

5. LEMMA 2. *Let $f(z) \in B$ and $0 < \lambda \leqq 1$. $f(z) \in J_1(\lambda, E) \Leftrightarrow$ for no $h(z) \in B$ the inequality*

$$(1)\qquad |F(z)-f(z)+h(z)| \overset{*}{<} \frac{\lambda|F(z)-f(z)-h(z)|}{1+\sqrt{1-\lambda^2}}$$

holds throughout E.

Proof. If $\lambda = 1$, and if for some $g(z) \in B$,

$$(2)\qquad |F(z)-g(z)| \overset{*}{<} |F(z)-f(z)|$$

throughout E, then let $h(z) \equiv f(z) - g(z)$. If, for $z_0 \in E$, $f(z_0) = F(z_0)$ then $g(z_0) = F(z_0)$, hence $h(z_0) = 0$ which implies (1) for z_0. If $f(z_0) \neq F(z_0)$, then

$$|F(z_0)-f(z_0)-h(z_0)| = |F(z_0)-2f(z_0)+g(z_0)| = |F(z_0)-f(z_0)|\left|2-\left(\frac{F(z_0)-g(z_0)}{F(z_0)-f(z_0)}\right)\right|$$
$$\geq |F(z_0)-f(z_0)|\left(2-\left|\frac{F(z_0)-g(z_0)}{F(z_0)-f(z_0)}\right|\right) > |F(z_0)-f(z_0)| > |F(z_0)-g(z_0)|$$
$$= |F(z_0)-f(z_0)+h(z_0)|,$$

thus (1) holds for every $z \in E$.
Conversely, (1) (with $\lambda = 1$) implies that when $F(z_0) \neq f(z_0)$, $\mathrm{Re}\left(\frac{h(z_0)}{f(z_0)-F(z_0)}\right) > 0$. Let

$$a = \sup\left\{\left|\frac{h(z)}{f(z)-F(z)}\right|^2 \Big/ 2\mathrm{Re}\left(\frac{h(z)}{f(z)-F(z)}\right); z \in E, f(z) \neq F(z)\right\},$$

and put $g(z) \equiv f(z) - \dfrac{h(z)}{1+a}$, then inequality (2) easily follows. When, for $z_0 \in E$, $F(z_0) = f(z_0)$, then $h(z_0) = 0$, hence $g(z_0) = F(z_0)$, and again (2) is satisfied.

If $\lambda < 1$, let $h(z) \equiv \dfrac{1}{\sqrt{1-\lambda^2}}(f(z)-g(z))$ be the relation between $g(z) \in B$ and $h(z) \in B$. Obviously,

$$|F(z_0)-g(z_0)| \overset{*}{<} \lambda|F(z_0)-f(z_0)| \Leftrightarrow \text{(1) holds for } z_0.$$

Note that Lemma 2 holds if $0 < \lambda < 1$ and we replace above "$J_1(\lambda, E)$" with "$J_2(\lambda, E)$", and replace "$\overset{*}{<}$" with "$\leqq$".

LEMMA 3. *Let $f(z) \in B$, $E_0 = \{z \in E; f(z) = F(z)\}$. For every $E' \subseteq E - E_0$, define the set $R(E')$ in the Euclidean space ε^{2n}:*

$$R(E') = \left\{(r_1, s_1, r_2, s_2, \cdots, r_n, s_n);\ r_i = \mathrm{Re}\left[\frac{f_i(z)}{f(z)-F(z)}\right],\ s_i = \mathrm{Im}\left[\frac{f_i(z)}{f(z)-F(z)}\right],\ 1 \leqq i \leqq n,\ z \in E'\right\}.$$

For any $a = (a_1, b_1, \cdots, a_n, b_n) \in \varepsilon^{2n}$ let $\pi(a, 1) = \{x \in \varepsilon^{2n}; (x, a) > 0\}$, and for $0 < \mu < 1$ let

$$\pi(\boldsymbol{a},\mu) = \{\boldsymbol{x}=(x_1,y_1,\cdots,x_n,y_n)\in\varepsilon^{2n};\left|\sum_{k=1}^{n}(x_k+iy_k)(a_k-ib_k)-1\right|<\mu\}.$$

Given $0<\lambda\leqq 1$, $f(z)\in J_1(\lambda,E)\Leftrightarrow$ *for every* $\boldsymbol{a}=(a_1,b_1,\cdots,a_n,b_n)\in\varepsilon^{2n}$ *which satisfies* $\sum_{k=1}^{n}(a_k-ib_k)f_k(z)=0$ *on* $E_0, R(E-E_0)\nsubseteq\pi(\boldsymbol{a},\lambda)$.

Proof. $\Leftarrow$: Suppose $f(z)\notin J_1(\lambda,E)$, by Lemma 2 there exists

$$h(z)\equiv\sum_{k=1}^{n}(a_k-ib_k)f_k(z)$$

which satisfies (1) on E, let $\boldsymbol{a}=(a_1,b_1,\cdots,a_n,b_n)$. If $\lambda=1$, (1) means: $h(z)=0$ on E_0, and $\mathrm{Re}\left[\frac{h(z)}{f(z)-F(z)}\right]>0$ on $E-E_0$ that is $R(E-E_0)\subseteq\pi(\boldsymbol{a},1)$. If $\lambda<1$, (1) means: $h(z)=0$ on E_0, and $\left|\sqrt{1-\lambda^2}\,\frac{h(z)}{f(z)-F(z)}-1\right|<\lambda$ on $E-E_0$ that is $R(E-E_0)\subseteq\pi(\boldsymbol{a}',\lambda)$ where $\boldsymbol{a}'=\sqrt{1-\lambda^2}\,\boldsymbol{a}$.

$\Rightarrow$: This is proved similarly by working backwards.

THEOREM 2. *Given* $0<\lambda\leqq 1$ *and* $f(z)\in B$, *suppose* $E_0=\{c_1,c_2,\cdots,c_l\}=$ $=\{z\in E;\ f(z)=F(z)\}$ *where* $0\leqq l<n$. $f(z)\in J_1(\lambda,E)\Leftrightarrow$ *there is a set* $E_{m+1}\subseteq E-E_0$ *of* $m+1$ *points,* $n-l\leqq m\leqq 2(n-l)$, *such that*

$$f(z)\in J_1(\lambda,E_{m+1}\cup E_0).$$

Proof. $\Leftarrow$: Obvious. $\Rightarrow$: Assuming to the contrary $f(z)\notin J_1(\lambda,E_{2n-2l+1}\cup E_0)$ for every $E_{2n-2l+1}=\{z_0,z_1,\cdots,z_{2n-2l}\}\subseteq E-E_0$, there is by Lemma 3 $\boldsymbol{a}=(a_1,b_1,\cdots,a_n,b_n)\in\varepsilon^{2n}$ such that $\sum_{k=1}^{n}(a_k-ib_k)f_k(z)=0$ on E_0 and $R(E_{2n-2l+1})\subseteq\pi(\boldsymbol{a},\lambda)$. Let $\boldsymbol{x}_j=(\mathrm{Re}f_1(c_j),\ \mathrm{Im}f_1(c_j),\cdots,\mathrm{Re}f_n(c_j),\ \mathrm{Im}f_n(c_j))$, $\boldsymbol{y}_j=(\mathrm{Im}f_1(c_j),-\mathrm{Re}f_1(c_j),\cdots,\mathrm{Im}f_n(c_j),-\mathrm{Re}f_n(c_j)),j=1,2,\cdots,l$, and for any $\boldsymbol{b}\in\varepsilon^{2n}$ let $H(\boldsymbol{b})=\{\boldsymbol{x};(\boldsymbol{b},\boldsymbol{x})=0\}$. Since the vectors $\boldsymbol{x}_1,\boldsymbol{y}_1,\cdots,\boldsymbol{x}_l,\boldsymbol{y}_l$ are linearly independent, $H=\bigcap_{j=1}^{l}H(\boldsymbol{x}_j)\cap H(\boldsymbol{y}_j)$ is $2n-2l$ dimensional. Let $\boldsymbol{x}(z)\in R(E-E_0)$ be the point associated with $z\in E-E_0$. By our assumption $\boldsymbol{a}\in H\bigcap_{i=0}^{2n-2l}\pi(\boldsymbol{x}(z_i),\lambda)$. Using the well known Helly's theorem on the intersection of convex sets in ε^{2n-2l} (the sets $\pi(\boldsymbol{x}(z),\lambda)$ are convex), there exists $\boldsymbol{a}'=(a_1',b_1',\cdots,a_n',b_n')$ in $H\bigcap_{z\in E-E_0}\pi(\boldsymbol{x}(z),\lambda)$, that is $\sum_{k=1}^{n}(a_k'-ib_k')f_k(z)=0$ on E_0 and $R(E-E_0)\subseteq\pi(\boldsymbol{a}',\lambda)$, meaning by Lemma 3 $f(z)\notin J_1(\lambda,E)$ — a contradiction! We proved the existence of E_{m+1} with $m\leqq 2\ (n-l)$, clearly $m\geqq n-l$, since otherwise, by condition Γ, there exists $g(z)\in B$ equal to $F(z)$ on $E_{m+1}\cup E_0$.

REMARKS. a_1) Obviously if E_0 has n points at least, then $f(z)\in J_1(\lambda,E_0')$ where $E_0'(\subseteq E_0)$ contains n or more points.

b_1) A similar theorem holds for $J_2(\lambda,E)$.

c_1) If $F(z),f(z),f_k(z)$ $(1\leqq k\leqq n)$ are real we obtain $m=n-l$ (since in the proof $R(E-E_0)$ can be embedded in ε^n). This is also true for $J_2(\lambda,E)$.

6. The following theorem is established in [2] for the case $f_k(z) \equiv z^{n-k}$ $(1 \leqq k \leqq n)$ and $E_0 = \emptyset$.

THEOREM 3. *Suppose* $f(z) \in B$, $E_0 = \{c_1, \cdots, c_l\} = \{z \in E;\ f(z) = F(z)\}$ *and* $0 \leqq l < n$.

(i) $f(z) \in J_1(1, E) \Rightarrow$ *there exist* — a set $E_{m+1} = \{z_0, \cdots, z_m\} \subseteq E - E_0$ *where* $n - l \leqq m \leqq 2(n-l)$, *constants* $\lambda_i > 0$ $(0 \leqq i \leqq m)$ *and* w_j $(1 \leqq j \leqq l)$, *such that*

$$\sum_{i=0}^{m} \lambda_i f_k(z_i)\overline{(F(z_i) - f(z_i))} + \sum_{j=1}^{l} w_j f_k(c_j) = 0 \tag{3}$$

for every $1 \leqq k \leqq n$.

(ii) *If* (3) *holds for some such* E_{m+1}, λ_i, w_j, *then* $f(z) \in J_1(1, E_{m+1} \cup E_0)$.

Proof. We retain the notations of Theorem 2. (i) By Theorem 2, $f(z) \in J_1(1, E_{m+1} \cup E_0)$ for some such E_{m+1}. By Lemma 3, $\boldsymbol{a} \in H \Rightarrow R(E_{m+1}) \nsubseteq \pi(\boldsymbol{a}, 1)$, that is $CR(E_{m+1}) \cap H(\boldsymbol{a}) \neq \emptyset$, meaning $CR(E_{m+1}) \cap H^{\perp} \neq \emptyset$ (where $CR(E_{m+1})$ denotes the convex hull of $R(E_{m+1})$, and $H^{\perp}$ the space orthogonal to H). Therefore there exist constants $\mu_i \geqq 0$ $\sum_{i=0}^{m} \mu_i = 1$, real constants α_j, β_j $(1 \leqq j \leqq l)$ such that $\sum_{i=0}^{m} \mu_i x(z_i) = \sum_{j=1}^{l} \alpha_j x_j + \beta_j y_j$. Putting $w_j = \alpha_j - i\beta_j$, $\lambda_i = \mu_i |f(z_i) - F(z_i)|^{-2}$, we obtain (3). At least $n - l$ constants λ_i are positive, since (3) with $m < n - l$ gives (due to condition Γ) $\lambda_i = 0$ and $w_j = 0$ which contradicts $\sum_{i=0}^{m} \lambda_i > 0$.

(ii) Conversely, (3) means $CR(E_{m+1}) \cap H^{\perp} \neq \emptyset$, that is

$$\boldsymbol{a} \in H \ \Rightarrow\ R(E_{m+1}) \ \nsubseteq\ \pi(\boldsymbol{a}, 1),$$

and by Lemma 3 $f(z) \in J_1(1, E_{m+1} \cup E_0)$.

REMARK. Theorem 3 with $f_k(z) \equiv z^{n-k}$, $F(z) \equiv z^n$ and $E_0 = \emptyset$ supplies us with the structure of an infrapolynomial [1]: let $p(z) \equiv z^n + a_1 z^{n-1} + \cdots + a_n \neq 0$ on E. $p(z)$ is an infrapolynomial on $E \Leftrightarrow$ there exists $E_{m+1} = \{z_0, z_1, \cdots, z_m\} \subseteq E$ with $n \leqq m \leqq 2n$ for which $p(z)$ is an infrapolynomial $\Leftrightarrow$ there exist $\lambda_i > 0$ $i = 0, 1, \cdots, m$ $\sum_{i=0}^{m} \lambda_i = 1$ such that $\sum_{i=0}^{m} \lambda_i \prod_{j=0;j\neq i}^{m} (z - z_j)$ is divisible by $p(z)$.

7. We shall find in this section the structure of $f(z) \in J_3(\lambda, E)$ when $F(z), f_k(z), f(z)$ are real and $\lambda < 1$.

Given $0 < \lambda \leqq 1$, an integer n, and a set E, denote by $I_1(\lambda, n, E)$ $[I_2(\lambda, n, E)]$ the set of all polynomials $p(z) \equiv z^n + a_1 z^{n-1} + \cdots + a_n$ with the property — for no polynomial $q(z) \equiv z^n + \cdots$ the inequality $|q(z)| \overset{*}{<} \lambda |p(z)|$ $[|q(z)| \leqq \lambda |p(z)|]$ holds throughout E. Let $I_3(\lambda, n, E) = I_1(\lambda, n, E) - I_2(\lambda, n, E)$.

REMARKS. a_2) It is shown in [3] that when E is compact $I_3(1, n, E) = \emptyset$ (compare with Lemma 1), also by similar methods to Theorem 1

$I_2(1,n,E)\bigcup_{\lambda<1} I_3(\lambda,n,E)$ consists of every polynomial of the form $z^n + a_1 z^{n-1} + \cdots + a_n$ (for finite E).

b$_2$) Given $z_0 \in E$, $t(z) \in I_k(\lambda,n,E)$ and $t(z_0)=0 \Leftrightarrow \dfrac{t(z)}{z-z_0} \in I_k(\lambda,n-1,E-\{z_0\})$.

c$_2$) Presently we shall need the following result (see [4]): If $F(z)$, $f_k(z)$ $(1 \leqq k \leqq n)$ are defined on $E=\{z_0,z_1,\cdots,z_n\}$, and if condition Γ holds and $F(z) \notin B$, let $f^{(i)}(z) \in B$ be the function for which $F(z)-f^{(i)}(z)$ vanishes on $E-\{z_i\}$, $i=0,1,\cdots,n$. Given any $f(z)\in B$ there is a polynomial $p(z) \equiv z^n + \cdots$ (or alternately, given any $p(z)\equiv z^n+\cdots$ there is $f(z)\in B$) such that

$$p(z_i) \Big/ \prod_{\substack{j=0\\ j\neq i}}^{n} (z_i - z_j) = (F(z_i) - f(z_i))/(F(z_i) - f^{(i)}(z_i)), \quad 0 \leqq i \leqq n. \tag{4}$$

d$_2$) When $f(z)$ and $p(z)$ are related by (4) on such a set E,

$$f(z) \in J_k(\lambda,E) \Leftrightarrow p(z) \in I_k(\lambda,n,E).$$

$$I_1(1,n,E) = \left\{ \sum_{i=0}^{n} \lambda_i \prod_{\substack{j=0\\ j\neq i}}^{n} (z-z_j);\ \lambda_i \geqq 0,\ \sum_{i=0}^{n} \lambda_i = 1 \right\}$$

so it follows from (4) that

$$J_1(1,E) = \{f(z)\in B;\ \lambda_i = (F(z_i)-f(z_i))/(F(z_i)-f^{(i)}(z_i)), \lambda_i \geqq 0,\ \sum_{i=0}^{n} \lambda_i = 1\}.$$

e$_2$) When $F(z), f_k(z)$ $(1\leqq k \leqq n)$ are real over $E=\{z_0,z_1,\cdots,z_m\}$ $(m \geqq n)$, clearly $J_1(1,E)$ consists of real functions only. By c$_1$ $f(z)\in J_1(1,E_{n+1})$ for some $E_{n+1}=\{z'_0,\cdots,z'_n\} \subseteq E$, and d$_2$ yields a simpler representation than that in Theorem 3.

From now on we limit ourselves to the real case only, that is, E is a finite point set in the real line, $F(x), f_k(x)$ $(1 \leqq k \leqq n)$ are real on E and we restrict B to real functions only. This is necessary as the following results appear to have no simple equivalent generalizations for the complex case.

THEOREM 4. *Let $F(x)\notin B$, and suppose $f(x)\in B$ is equal to $F(x)$ on n points at most. For any $E_{n+1}=\{x_0,x_1,\cdots,x_n\} \subseteq E$, let*

$$\lambda(E_{n+1}) = \left[\sum_{i=0}^{n} \left| \frac{F(x_i)-f(x_i)}{F(x_i)-f^{(i)}(x_i)} \right| \right]^{-1}$$

where $f^{(i)}(x)$ is defined for E_{n+1} as in c$_2$ (define $\lambda(E_{n+1})=0$ if $F(x_{i'})=f^{(i')}(x_{i'})$ for some (i'). Let $\lambda'=\sup\{\lambda(E_{n+1}); E_{n+1}\subseteq E\}$, then $0<\lambda'\leqq 1$ and

(i) $\lambda' = 1 \Leftrightarrow f(x) \in J_1(1, E)$,
(ii) $\lambda' < 1 \Leftrightarrow f(x) \in J_3(\lambda', E)$.

Proof. Assume $\lambda' > 1$ and suppose $\lambda' = \lambda(E'_{n+1})$ where $E'_{n+1} = \{x'_0, x'_1, \cdots, x'_n\}$. $F(x) \not\equiv \sum_{j=1}^n a_j f_j(x)$ on E'_{n+1}, since otherwise $\lambda(E'_{n+1}) = 0$. Let $p(x) \equiv x^n + b_0 x^{n-1} + \cdots + b_n$ be related to $f(x)$ by (4) for the set E'_{n+1}. Let $\omega(x) \equiv \prod_{i=0}^n (x - x'_i)$ then

$$(5) \quad 1 = \sum_{i=0}^{n} \frac{p(x'_i)}{\omega'(x'_i)} \leqq \sum_{i=0}^{n} \left| \frac{p(x'_i)}{\omega'(x'_i)} \right| = \sum_{i=0}^{n} \left| \frac{F(x'_i) - f(x'_i)}{F(x'_i) - f^{(i)}(x'_i)} \right| = \frac{1}{\lambda(E'_{n+1})}$$

which contradicts $\lambda(E'_{n+1}) > 1$. $\lambda' > 0$, since $\lambda' = 0 \Rightarrow F(x) \in B$!

(i) $\Rightarrow$: Let $\lambda' = \lambda(E'_{n+1}) = 1$, using (4) and (5) we have $p(x'_i)/\omega'(x'_i) \geqq 0$, and from d_2 $f(x) \in J_1(1, E'_{n+1}) \subseteq J_1(1, E)$.

(i) $\Leftarrow$: By c_1 there exists E'_{n+1} such that $f(x) \in J_1(1, E'_{n+1})$, using (4) and d_2, (5) follows with equality everywhere, giving $\lambda(E'_{n+1}) = 1$.

(ii) $\Rightarrow$: Suppose $\lambda' = \lambda(E'_{n+1})$ and $f(x) \notin J_1(\lambda', E'_{n+1})$, then there exists $g(x) \in B$ such that $|g(x'_i) - F(x'_i)| \overset{*}{<} \lambda' |f(x'_i) - F(x'_i)|$ $i = 0, 1, \cdots, n$. Let (4) relate $r(x) \equiv x^n + \cdots$ and $g(x)$ for E'_{n+1}, then

$$1 = \sum_{i=0}^{n} \frac{r(x'_i)}{\omega'(x'_i)} \leqq \sum_{i=0}^{n} \left| \frac{r(x'_i)}{\omega'(x'_i)} \right| =$$

$$= \sum_{i=0}^{n} \left| \frac{F(x'_i) - g(x'_i)}{F(x'_i) - f^{(i)}(x'_i)} \right| < \lambda(E'_{n+1}) \sum_{i=0}^{n} \left| \frac{F(x'_i) - f(x'_i)}{F(x'_i) - f^{(i)}(x'_i)} \right| = 1,$$

this contradiction implies $f(x) \in J_1(\lambda', E'_{n+1}) \subseteq J_1(\lambda', E)$.

For any $E_{n+1} = \{x_0, \cdots, x_n\}$, $f(x) \notin J_2(\lambda(E_{n+1}), E_{n+1})$. This is obvious if $\lambda(E_{n+1}) = 0$. If $\lambda(E_{n+1}) > 0$, let (4) relate $p(x) \equiv x^n + \cdots$ and $f(x)$ on E_{n+1}, define

$$r(x) \equiv \sum_{i=0}^{n} \lambda(E_{n+1}) \left| \frac{p(x_i)}{\omega'(x_i)} \right| \frac{\omega(x)}{x - x_i}, \text{ where } \omega(x) \equiv \prod_{i=0}^{n} (x - x_i)$$

then $|r(x_i)| = \lambda(E_{n+1}) |p(x_i)|$, meaning $f(x) \notin J_2(\lambda(E_{n+1}), E_{n+1})$ $(\supseteq J_2(\lambda', E_{n+1}))$. Let

$$C(x) = \{(a_1, \cdots, a_n) \in \varepsilon^n;\ \left| \sum_{i=1}^{n} a_i f_i(x) - F(x) \right| \leqq \lambda' |f(x) - F(x)| \},$$

we have $\bigcap_{i=0}^{n} C(x_i) \neq \varnothing$ for every $E_{n+1} \subseteq E$, and by Helly's theorem (the sets $C(x)$ are convex) $\bigcap_{x \in E} C(x) \neq \varnothing$, so $f(x) \notin J_2(\lambda', E)$. Therefore $f(x) \in J_3(\lambda', E)$.

(ii) $\Leftarrow$: $J_1(1, E) = J_2(1, E)$, therefore $\lambda' < 1$.

COROLLARY 1. *Suppose* $f(x) = F(x)$ *only on* $E_k = \{x_0, x_1, \cdots, x_{k-1}\}$ *where* $0 \leqq k < n$, *let* $0 < \lambda < 1$.

(i) $f(x) \in J_3(\lambda, E) \Rightarrow$ *there exist*: $E_{n+1} = E_k \cup \{x_k, x_{k+1}, \cdots, x_n\}$, *constants*

$\alpha_j,\beta_j \geqq 0$ $(j=k,k+1,\cdots,n)$ with $\Sigma_{j=k}^n \alpha_j = \Sigma_{j=k}^n \beta_j = 1$ and $\alpha_j\beta_j = 0$ such that

$$(6) \qquad f(x_j) - F(x_j) = \frac{(1+\lambda)\alpha_j - (1-\lambda)\beta_j}{2\lambda}(f^{(j)}(x_j) - F(x_j)) \ (k \leqq j \leqq n).$$

(ii) *If (6) is satisfied for such E_{n+1}, α_j, β_j then $f(x) \in J_3(\lambda, E_{n+1})$.*

Proof. (i) The existence of such E_{n+1} for which $f(x) \in J_3(\lambda, E_{n+1})$ is assured by c_1, and by Theorem 4: $\lambda = \left[\sum_{i=0}^n \left| \frac{F(x_i) - f(x_i)}{F(x_i) - f^{(i)}(x_i)} \right| \right]^{-1}$. Define $\alpha_j = \frac{2\lambda}{1+\lambda}\left[\frac{F(x_j) - f(x_j)}{F(x_j) - f^{(j)}(x_j)} \right]$ when this ratio is positive, and when it is negative define $\beta_j = -\frac{2\lambda}{1-\lambda}\left[\frac{F(x_j) - f(x_j)}{F(x_j) - f^{(j)}(x_j)} \right]$ (take $\alpha_j = 0$ $[\beta_j = 0]$ when $\beta_j \neq 0$ $\cdot[\alpha_j \neq 0]$), representation (6) then follows.

(ii) Calculating $\lambda(E_{n+1})$ of Theorem 4 from (6) we have $\lambda(E_{n+1}) = \lambda < 1$, the rest follows from Theorem 4(ii).

8. We aim to prove here a supplement to Corollary 1:

COROLLARY 2. *With the assumptions of Corollary 1,*

(i) *$f(x) \in J_2(\lambda, E) \Rightarrow$ there exist: $E_{n+1} = E_k \cup \{x_k, x_{k+1}, \cdots, x_n\}$, constants $\lambda_{ij} > 0$ $(k \leqq i \neq j \leqq n)$ $\quad \Sigma\lambda_{ij} = 1$ such that*

$$(7) \quad f(x_j) - F(x_j) = \sum_{\substack{i=k \\ i \neq j}}^n \frac{(1+\lambda)\lambda_{ij} - (1-\lambda)\lambda_{ji}}{2\lambda}(f^{(j)}(x_j) - F(x_j)) \ (k \leqq j \leqq n)$$

(ii) *If (7) is satisfied for such E_{n+1}, λ_{ij}, then $f(x) \in J_2(\lambda, E_{n+1})$.*

The proof of Corollary 2 depends on the following two lemmas of which the first is obvious:

LEMMA 4. *If $C \subseteq \varepsilon^n$ is a compact convex set, let $\pi(\boldsymbol{a},\lambda) = \{\boldsymbol{x} \in \varepsilon^n; |(\boldsymbol{x},\boldsymbol{a}) - 1| \leqq \lambda\}$ where $0 < \lambda < 1$ and $\boldsymbol{a} \in \varepsilon^n$. $C \nsubseteq \pi(\boldsymbol{a},\lambda)$ for every $\boldsymbol{a} \in \varepsilon^n \Leftrightarrow$ the orthogonal projection of C on any line l through O is a closed interval $[(\boldsymbol{x}(l),\boldsymbol{y}(l)]$ with the property: either $O \in [\boldsymbol{x}(l),\boldsymbol{y}(l)]$ or $\frac{d(\boldsymbol{y}(l),O)}{d(\boldsymbol{x}(l),O)} > \frac{1+\lambda}{1-\lambda}$* ($d$ denotes the Euclidean metric function).

LEMMA 5. *Let $A = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\} \subseteq \varepsilon^n$ be an affine independent set. For $0 < \lambda < 1$ and $\pi(\boldsymbol{a},\lambda)$ as above, define $\boldsymbol{x}_{ij} = \frac{(1+\lambda)\boldsymbol{x}_j - (1-\lambda)\boldsymbol{x}_i}{2\lambda}$ $(0 \leqq i \neq j \leqq n)$, let $F = \{\boldsymbol{x}_{ij}; 0 \leqq i \neq j \leqq n\}$ and $C(F)$ its convex hull. Then, $O \in \operatorname{int} C(F) \Leftrightarrow A \nsubseteq \pi(\boldsymbol{a},\lambda)$ for every $\boldsymbol{a} \in \varepsilon^n$.*

Proof. For any $G \subseteq \varepsilon^n$ denote by $C(G)$ its convex hull.

STATEMENT a: Let $F' = \{x_{ij}; 0 \leqq i \neq j < n\}$, the relative interior of $C(F')$ is contained in $\operatorname{int} C(F)$. This is due to x_{n0} and x_{0n} being strictly separated by π — the plane containing F' —whence $\operatorname{relint} C(F') \subseteq \operatorname{int} C(F' \cup \{x_{n0}, x_{0n}\}) \subseteq \operatorname{int} C(F)$. Moreover $A \subseteq \operatorname{int} C(F)$.

STATEMENT b: Let π_1, π_2 be parallel distinct planes supporting $C(A)$ at x_j, x_i respectively. Let π_3 support $C(F)$ so that π_1 separates π_3 and π_2, then $x_{ij} \in \pi_3$. This is easily verified for $n \leqq 3$. Suppose $n > 3$, let $x_{kl} \in C(F) \cap \pi_3$ and assume that $A' = \{x_0, x_1, \cdots, x_{n-1}\}$ contains x_i, x_j, x_k, x_l, let the plane π contain A', and let $F' = \{x_{ij}; 0 \leqq i \neq j < n\}$, $\pi'_r = \pi_r \cap \pi$ $r = 1,2,3$. Reducing the problem to the $(n-1)$ dimensional "space" π, the proof is carried out by induction.

STATEMENT c: With the notations of statement b,

$$\frac{d(\pi_3, \pi_2)}{d(\pi_3, \pi_1)} = \frac{d(x_{ij}, x_i)}{d(x_{ij}, x_j)} = \frac{1+\lambda}{1-\lambda}.$$

$\Leftarrow$: Assume $O \notin \operatorname{int} C(F)$. Take π_r $(r = 1,2,3)$ as in statement b so that π_3 separates O from $C(F)$ and $C(F) \cap \pi_3$ is an $n-1$ dimensional face of $C(F)$. Pass a line l orthogonal to π_3 through O, let $l \cap \pi_r = x'_r$ $(r = 1,2,3)$, then $[x'_1, x'_2]$ is the orthogonal projection of $C(A)$ on l, and by statement c

$$\frac{d(\pi_3, \pi_2)}{d(\pi_3, \pi_1)} = \frac{1+\lambda}{1-\lambda} = \frac{d(x'_3, x'_2)}{d(x'_3, x'_1)} \geqq \frac{d(O, x'_2)}{d(O, x'_1)}$$

which contradicts Lemma 4.

$\Rightarrow$: Let l be any line through O, let π_r $(r = 1,2,3)$ be as in statement b and orthogonal to l, let $x'_r = l \cap \pi_r$ $(r = 1,2,3)$ and suppose $O \in (x'_1, x'_3)$, then $\dfrac{d(O, x'_2)}{d(O, x'_1)} > \dfrac{d(x'_3, x'_2)}{d(x'_3, x'_1)}$ (by c) $= \dfrac{1+\lambda}{1-\lambda}$, and according to Lemma 4,

$$a \in \varepsilon^n \Rightarrow \pi(a, \lambda) \not\supseteq C(A).$$

Proof of Corollary 2. (ii)

$$\frac{1}{\lambda(E_{n+1})} = \sum_{i=0}^{n} \left| \frac{F(x_i) - f(x_i)}{F(x_i) - f^{(i)}(x_i)} \right| =$$

$$= \sum_{j=k}^{n} \left| \sum_{\substack{i=k \\ i \neq j}}^{n} \frac{(1+\lambda)\lambda_{ij} - (1-\lambda)\lambda_{ji}}{2\lambda} \right| < \sum_{j=k}^{n} \sum_{\substack{i=k \\ i \neq j}}^{n} \frac{(1+\lambda)\lambda_{ij} + (1-\lambda)\lambda_{ji}}{2\lambda} = \frac{1}{\lambda},$$

Theorem 4 gives $f(x) \in J_1(\lambda(E_{n+1}), E_{n+1}) \subseteq J_2(\lambda, E_{n+1})$.

(i) By Remark c_1 $f(x) \in J_2(\lambda, E_{n+1})$ for a suitable $E_{n+1} \supseteq E_k$. Let (4) relate $p_n(x) \equiv x^n + \cdots$ and $f(x)$ for E_{n+1}, let $p_{n-k}(x) \equiv p_n(x) / \prod_{j=0}^{k-1}(x - x_j)$, by d_2 and b_2 $p_{n-k}(x) \in I_2(\lambda, n-k, E_{n+1} - E_k)$. Let $x_i = (p_{n-k}(x_i))^{-1}(x_i^{n-k-1}, \cdots, x_i, 1) \in \varepsilon^{n-k}$ $i = k, k+1, \cdots, n$, and let $\sigma = \{x_k, x_{k+1}, \cdots, x_n\}$. As in Lemma 3:

$$p_{n-k}(x) \in I_2(\lambda, n-k, E_{n+1}-E_k) \Leftrightarrow \sigma \nsubseteq \pi(a,\lambda) = \{x \in \varepsilon^{n-k}; |(a,x)-1| \leqq \lambda\}$$

for every $a \in \varepsilon^{n-k}$.

σ is $(n-k)$-dimensional, for assume it is not so, then the linear hull of σ, $L(\sigma$ contains the origin; since otherwise the orthogonal projection of $L(\sigma)$ on a line through the origin and orthogonal to $L(\sigma)$ is a single point; and this by Lemma 4, contradicts the result: $\sigma \nsubseteq \pi(a,\lambda)$ for every $a \in \varepsilon^{n-k}$. Therefore, there exist $n-r$ $(\leqq n-k)$ points in σ, suppose $x_{r+1}, x_{r+2}, \cdots, x_n$, and real constants $a_{r+1}, a_{r+2}, \cdots, a_n$ not all zero, such that $\sum_{i=r+1}^n a_i x_i = 0$, but this equation has only one solution $a_{r+1} = a_{r+2} = \cdots = a_n = 0$, which is a contradiction.

Hence by Lemma 5 $O \in \operatorname{int} C(F)$ (we substitute σ for A and ε^{n-k} for ε^n in the lemma), so if $2\lambda x_{ij} = (1+\lambda)x_j - (1-\lambda)x_i$ $(k \leqq i \neq j \leqq n)$ then there are λ_{ij} as above for which $\sum \lambda_{ij} x_{ij} = O$; consideration of the coordinates and use of the Lagrange interpolation polynomial gives

$$p_{n-k}(x) \equiv \sum_{j=k}^{n} \left(\sum_{\substack{i=k \\ i \neq j}}^{n} \frac{(1+\lambda)\lambda_{ij} - (1-\lambda)\lambda_{ji}}{2\lambda} \right) \prod_{\substack{i=k \\ i \neq j}}^{n} (x - x_i),$$

(7) follows immediately.

9. For every $f(x) \equiv \sum_{i=1}^n a_i f(x) \in B$ we associate here the point $(a_1, a_2, \cdots, a_n) \in \varepsilon^n$, then

THEOREM 5. *Under the preliminary conditions of Theorem* 4, $J_1(\lambda, E)$ *is compact and connected.*

Proof. By c_1 $J_1(\lambda, E) = \cup J_1(\lambda, E_{n+1})$, it is sufficient therefore to establish compactness of $J_1(\lambda, E_{n+1})$: $E_{n+1} = \{x_0, \cdots, x_n\}$. According to Theorem 4,

$$\sum_{i=0}^{n} \left| \frac{F(x_i) - f(x_i)}{F(x_i) - f^{(i)}(x_i)} \right| \leqq \frac{1}{\lambda} \Leftrightarrow f(x) \in J_1(\lambda, E_{n+1}),$$

and compactness is now obvious.

Let $f(x) \in J_1(\lambda, E_{n+1})$ and consider the nontrivial case: $F(x) \not\equiv \sum_{k=1}^n a_k f_k(x)$ on E_{n+1}. Let $g_\alpha(x) \equiv \alpha f(x) + (1-\alpha) f^{(0)}(x)$ $(0 \leqq \alpha \leqq 1)$, then

$$\sum_{i=0}^{n} \left| \frac{F(x_i) - g_\alpha(x_i)}{F(x_i) - f^{(i)}(x_i)} \right| \leqq 1 - \alpha + \alpha \sum_{i=0}^{n} \left| \frac{F(x_i) - f(x_i)}{F(x_i) - f^{(i)}(x_i)} \right| \leqq 1 - \alpha + \frac{\alpha}{\lambda} \leqq \frac{1}{\lambda}$$

that is $g_\alpha(x) \in J_1(\lambda, E_{n+1})$.

We finish by showing that $J_1(1,E)$ is connected. Let $g_1(x)$, $g_2(x) \in J_1(1,E)$, there are $E_{n+1}^{(i)} = \{x_0^{(i)}, \cdots, x_n^{(i)}\}$ $(i = 1,2)$ such that $g_i(x) \in J_1(1, E_{n+1}^{(i)})$. Suppose now $E_{n+1}^{(1)} \cap E_{n+1}^{(2)} = \{x_1', x_2', \cdots, x_n'\}$, let $g(x) \in B$ be the function for which $F(x) - g(x)$ vanishes on $E_{n+1}^{(1)} \cap E_{n+1}^{(2)}$, then $g(x) \in J_1(1, E_{n+1}^{(i)})$ $(i = 1,2)$, and since $J_1(1, E_{n+1}^{(i)})$ is convex (deduced from d_2), the proof is established for this case. In general, construct sets $E_{n+1}^{(1)} = F_1, F_2, \cdots, F_r = E_{n+1}^{(2)}$ such that F_i $(1 \leqq i \leqq r)$

has $n+1$ points exactly, and $F_i \cap F_{i+1}$ has n points exactly, we may now connect $g_1(x), g_2(x)$ through the intermediate sets F_i.

REMARKS. Also it may be verified that $J_2(\lambda, E)$ $(0 < \lambda < 1)$ is open, connected, and its closure is $J_1(\lambda, E)$.

In [3] it is shown that $I_1(1, n, E)$ is convex if and only if $n = 0$ or $n = 1$ or E has $n+1$ points. This is not true in the general case as shown by the following counter example:

$$E = \{x_1, x_2, x_3, x_4\} \ (x_{i+1} > x_i), \ F(x_1) = F(x_2) = F(x_3) = 0 \text{ and } F(x_4) = 1,$$

and $f_1(x) \equiv x$, $f_2(x) \equiv 1$. We obtain from Remark d_2:

$$J_1(1, E) = \bigcup_{i=1}^{4} J_1(1, E - \{x_i\}) = J_1(1, E - \{x_2\})$$

which is convex.

REFERENCES

1. M. Fekete, *On the structure of extremal polynomials*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 95–103.

2. M. Fekete, *On the structure of polynomials of least deviation*, Bull. Res. Counc. Israel **5A** (1955), 11–19.

3. T. S. Motzkin and J. L. Walsh, *Underpolynomials and infrapolynomials*, Illinois J. Math. **1** (1957), 406–426.

4. T. S. Motzkin and J. L. Walsh, *On the derivative of a polynomial and Tchebycheff approximation*, Proc. Amer. Math. Soc. **4** (1953), 76–87.

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